A Posteriori Analysis of a Space-time Hybridizable Discontinuous Galerkin Method for the Advection-diffusion Problem

Yuan Wang

Applied Mathematics, University of Waterloo

Ontario Graduate Mathematics Conference

Outline

1. Motivation and preliminaries
    - Advection-diffusion in its space-time formulation
    - Advection-dominated regime
    - Hybridizable discontinuous Galerkin methods
2. A priori error analysis, briefly
    - The HDG method
    - Péclet-robust inf-sup conditions
    - A layer test case
3. A posteriori error analysis
    - Why adaptivity
    - Reliability, local efficiency and robustness
    - Main results
    - Two layer test cases

Time-dependent advection-diffusion problem

Standard formulation

$$\partial_t u + \overline{\nabla} \cdot (\bar{\beta} u) - \varepsilon \overline{\nabla}^2 u = f, \quad \text{in } \Omega(t), \quad 0 < t \leq T$$

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- Diffusion: $\varepsilon \overline{\nabla}^2 u$

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$$\Downarrow$$

$$\nabla \cdot (\beta u) - \varepsilon \overline{\nabla}^2 u = f, \quad \text{in } \mathcal{E}$$

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- Space-time advective field: $\beta := (1, \bar{\beta})$
- Space-time gradient: $\nabla := (\partial_t, \overline{\nabla})$
- Space-time advection: $\nabla \cdot (\beta u) = \partial_t u + \overline{\nabla} \cdot (\bar{\beta} u)$

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- Space-time advection: $\nabla \cdot (\beta u) = \partial_t u + \overline{\nabla} \cdot (\bar{\beta} u)$
- $(d+1)$-dimensional space-time domain: $\mathcal{E}$

Solution in the space-time formulation

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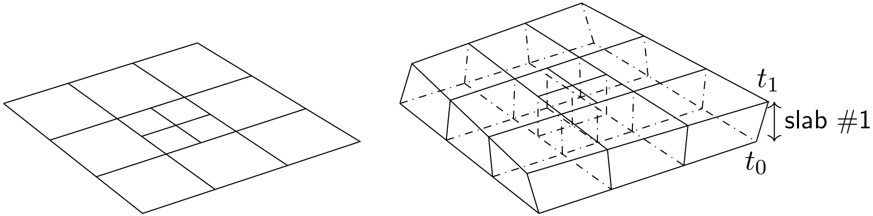

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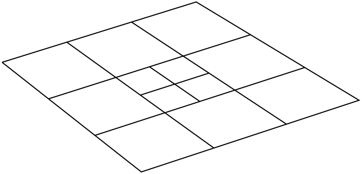

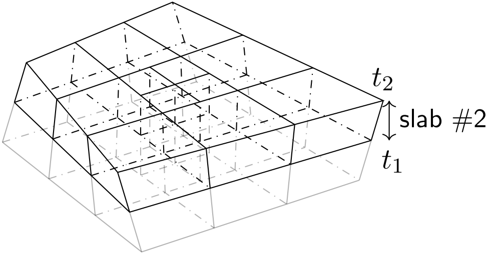

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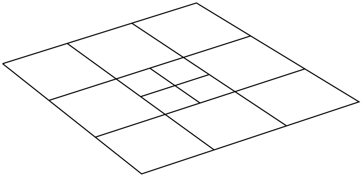

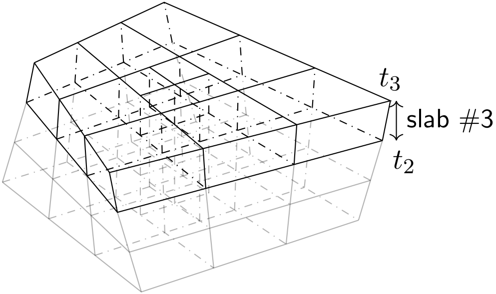

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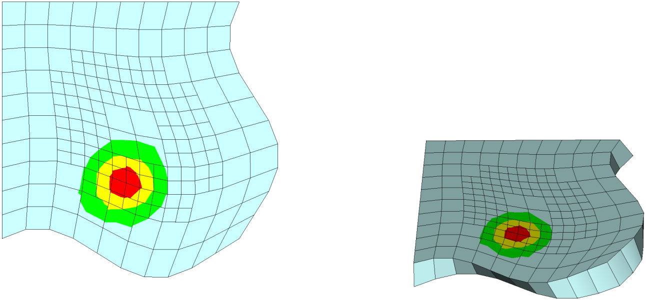
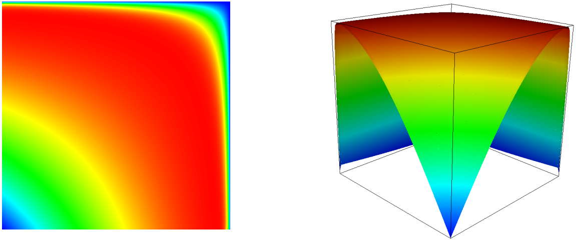


Why space-time

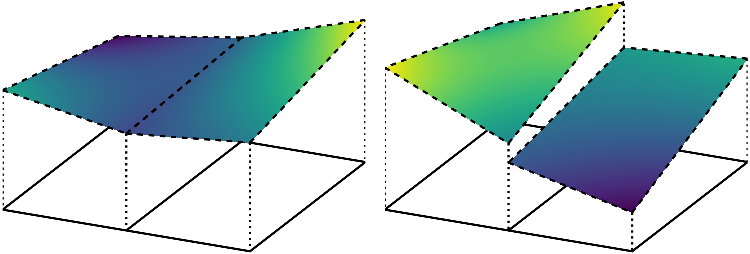
- Built-in domain movement
- Arbitrarily high-order approximation in both space and time
- Local adaptivity in both space and time (more on this later!)

Advection-dominated regime

- The focus of our work
  $0 < \varepsilon \ll 1$ and $\beta \sim \mathcal{O}(1) \Rightarrow$ Péclet number $\gg 1$
- Sharp boundary/interior layers



Discontinuous Galerkin methods



(a) CG: inter-element continuity (b) DG: discont. piecewise polynomial

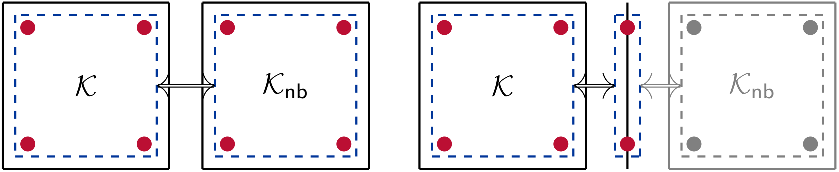
Why DG discretization

- Highly parallelizable implementations
- General meshes (hanging nodes, non-standard shapes)
- hp-adaptivity

Hybridizable discontinuous Galerkin methods

HDG = DG + Static condensation

- Stabilization properties of DG
- Advantages of DG due to its localized nature
- Reducing the number of globally coupled degrees-of-freedom



(a) DG dofs   (b) HDG dofs

Outline

A space-time HDG discretization

Find $\boldsymbol{u}_h \in \boldsymbol{V}_h$ such that

$$a_h\left(\boldsymbol{u}_h, \boldsymbol{v}_h\right) = \left(f, v_h\right)_{\mathcal{T}_h} + \langle g, \mu_h \rangle_{\partial\mathcal{E}_N} \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h$$

- $a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) := a_{h,d}(\boldsymbol{u}_h, \boldsymbol{v}_h) + a_{h,c}(\boldsymbol{u}_h, \boldsymbol{v}_h)$

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*G. Fu et al. (2015)

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- Diffusion:

$$a_{h,d}\left(\boldsymbol{u}, \boldsymbol{v}\right) := \left(\varepsilon \overline{\nabla} u, \overline{\nabla} v\right)_{\mathcal{T}_h} + \langle \varepsilon \alpha h_K^{-1} \left[\boldsymbol{u}\right], \left[\boldsymbol{v}\right] \rangle_{\mathcal{Q}_h} \\ - \langle \varepsilon \left[\boldsymbol{u}\right], \overline{\nabla}_{\bar{n}} v \rangle_{\mathcal{Q}_h} - \langle \varepsilon \overline{\nabla}_{\bar{n}} u, \left[\boldsymbol{v}\right] \rangle_{\mathcal{Q}_h}$$

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- Advection*:

$$a_{h,c}\left(\boldsymbol{u}, \boldsymbol{v}\right) := -\left(\beta u, \nabla v\right)_{\mathcal{T}_h} + \langle \zeta^{+} \beta \cdot n \lambda, \mu \rangle_{\partial\mathcal{E}_N} + \langle \left(\beta \cdot n\right) \lambda + \beta_s \left[\boldsymbol{u}\right], \left[\boldsymbol{v}\right] \rangle_{\partial\mathcal{T}_h}$$

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The main contribution

Notational setup:

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- δt: local time-step
- h: local spatial mesh size
- p_s, p_t are polynomial degrees in spatial and temporal directions
- The s-norm $|||\cdot|||_s$:

$$
\begin{aligned}
|||\boldsymbol{v}|||_v^2 := & \sum_{\mathcal{K}\in\mathcal{T}_h} \|v\|_{\mathcal{K}}^2 + \sum_{\mathcal{K}\in\mathcal{T}_h} \left\| \left|\beta_s - \tfrac{1}{2}\beta\cdot n\right|^{1/2} [\boldsymbol{v}] \right\|_{\partial\mathcal{K}}^2 + \sum_{F\in\partial\mathcal{E}_N} \left\| \left|\tfrac{1}{2}\beta\cdot n\right|^{1/2} \mu \right\|_F^2 \\
& + \sum_{\mathcal{K}\in\mathcal{T}_h} \varepsilon \|\overline{\nabla} v\|_{\mathcal{K}}^2 + \sum_{\mathcal{K}\in\mathcal{T}_h} \varepsilon h_K^{-1} \|[\boldsymbol{v}]\|_{\mathcal{Q}_{\mathcal{K}}}^2 \\
|||\boldsymbol{v}|||_s^2 := & |||\boldsymbol{v}|||_v^2 + \sum_{\mathcal{K}\in\mathcal{T}_h} \tau_\varepsilon \|\partial_t v\|_{\mathcal{K}}^2 \qquad (\tau_\varepsilon = \Delta t_{\mathcal{K}} \tilde{\varepsilon})
\end{aligned}
$$

The main contribution

A Péclet-robust a priori error analysis

- When $\varepsilon < \delta t = h$

$$|||\boldsymbol{u} - \boldsymbol{u}_h|||_s \leq c_T(h^{p_s+1/2} + \delta t^{p_t+1/2})$$

- When $\delta t = h < \varepsilon$

$$|||\boldsymbol{u} - \boldsymbol{u_h}|||_s \leq c_T(h^{p_s} + \delta t^{p_t})$$

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An $\mathbf{1/2}$ drop in the convergence rate is expected after mesh is sufficiently refined

Péclet-robust inf-sup conditions

Main inf-sup conditions:

Norm	**Nonrobust**		**Robust** (our contribution)
$\vert\vert\vert\cdot\vert\vert\vert_v$	$\vert\vert\vert w_h\vert\vert\vert_v^2 \lesssim \varepsilon^{-1} a_h(w_h, w_h)$	$\Longrightarrow$	$\vert\vert\vert w_h\vert\vert\vert_v \leq c_T \sup_{v_h \in V_h} \frac{a_h(w_h,v_h)}{\vert\vert\vert v_h\vert\vert\vert_v}$
			$\Big\Downarrow$
$\vert\vert\vert\cdot\vert\vert\vert_s$	$\vert\vert\vert w_h\vert\vert\vert_s \lesssim \varepsilon^{-1} \sup_{v_h \in V_h} \frac{a_h(w_h,v_h)}{\vert\vert\vert v_h\vert\vert\vert_s}$	$\Longrightarrow$	$\vert\vert\vert w_h\vert\vert\vert_s \leq c_T \sup_{v_h \in V_h} \frac{a_h(w_h,v_h)}{\vert\vert\vert v_h\vert\vert\vert_s}$

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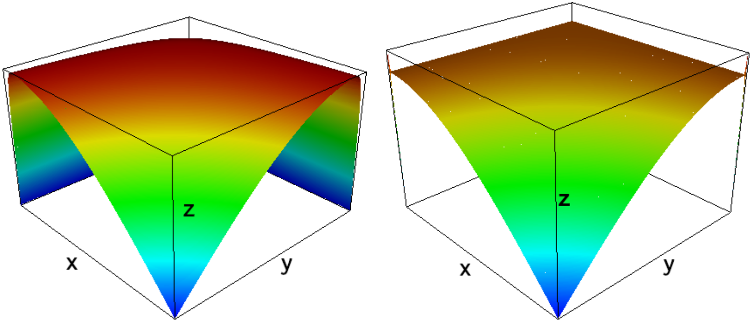
$\vert\vert\vert\cdot\vert\vert\vert_s$	$\vert\vert\vert w_h \vert\vert\vert_s \lesssim \varepsilon^{-1} \sup_{v_h \in V_h} \frac{a_h(w_h,v_h)}{\vert\vert\vert v_h \vert\vert\vert_s}$	$\Longrightarrow$	$\vert\vert\vert w_h \vert\vert\vert_s \leq c_T \sup_{v_h \in V_h} \frac{a_h(w_h,v_h)}{\vert\vert\vert v_h \vert\vert\vert_s}$

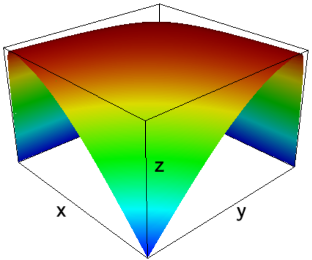
s-norm inf-sup stability proves instrumental in our a posteriori error analysis!

A boundary layer test case



(a) $\varepsilon = 10^{-2},\ T = 1,\ \delta t = h \approx 10^{-3}$ (b) $\varepsilon = 10^{-8},\ T = 1,\ \delta t = h \approx 10^{-3}$

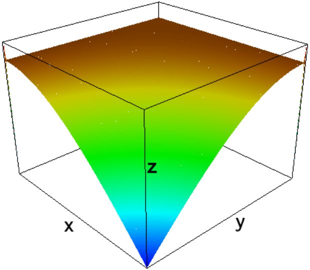
Convergence histories $(\varepsilon = 10^{-2})$



Cells per slab	Slabs	$p = 1$	Rates	$p = 2$	Rates	$p = 3$	Rates
100	10	4.0e-1	-	2.5e-1	-	1.3e-1	-
400	20	2.6e-1	0.6	1.1e-1	1.2	3.3e-2	2.0
1600	40	1.5e-1	0.8	3.5e-2	1.6	6.0e-3	2.5
6400	80	8.0e-2	0.9	9.7e-3	1.9	8.7e-4	2.8
25600	160	4.0e-2	1.0	2.5e-3	2.0	1.1e-4	3.0
102400	320	2.0e-2	1.0	6.2e-4	2.0	1.4e-5	3.0

Convergence histories
$(\varepsilon = 10^{-8})$

(error computed outside the layer)



Cells per slab	Slabs	$p = 1$	Rates	$p = 2$	Rates	$p = 3$	Rates
100	10	1.4e-2	-	7.0e-3	-	4.8e-3	-
400	20	1.3e-3	3.3	1.0e-5	9.4	5.3e-8	16.5
1600	40	4.4e-4	1.6	1.6e-6	2.7	3.6e-9	3.9
6400	80	1.5e-4	1.6	2.7e-7	2.6	2.8e-10	3.7
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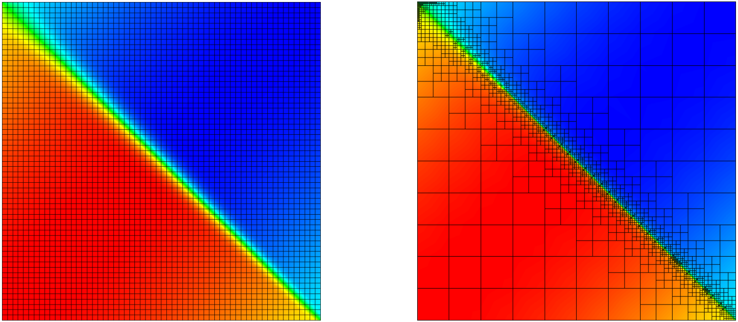
Why adaptivity?

Adaptive mesh refinement (AMR)

- Local errors in the narrow layer region dominate the global error
- Allocate more elements/dofs to areas with larger errors

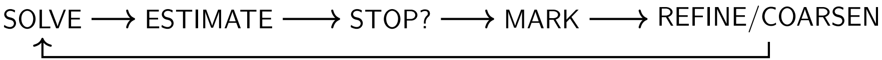
Let's see an example!

A motivating example



Element	L^2-error	Order	Element	L^2-err.	Order
1,000	1.0e-1	-	729	9.3e-5	-
8,000	8.2e-2	0.34	5,832	2.4e-5	2.02
64,000	6.6e-2	0.33	46,656	6.0e-6	2.01
512,000	5.2e-2	0.33	373,248	1.5e-6	2.00

The AMR loop



SOLVE ⟶ ESTIMATE ⟶ STOP? ⟶ MARK ⟶ REFINE/COARSEN

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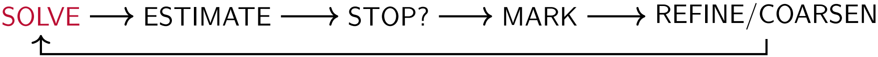

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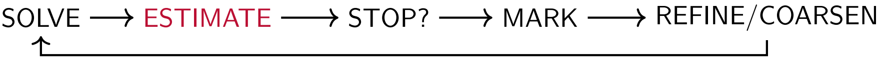

SOLVE ⟶ ESTIMATE ⟶ STOP? ⟶ MARK ⟶ REFINE/COARSEN

- Given an initial mesh, compute the HDG solution: $\boldsymbol{u}_h$
- Compute $\eta_{\mathcal{K}}$: the local error estimate on each element

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- $\left(\sum_{\mathcal{K}} \eta_{\mathcal{K}}^2\right)^{1/2} \leq \tau$?

Reliability:

$\|u - u_h\|_\Omega \leq c^* \left(\sum_{K \in \mathcal{T}} \eta_{\mathcal{K}}^2\right)^{1/2}$ with $c^* \sim \mathcal{O}(1)$ on all refinement levels

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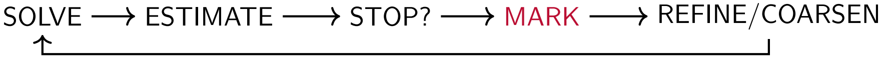
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- Mark elements with $\eta_{\mathcal{K}}$ bigger than a certain threshold

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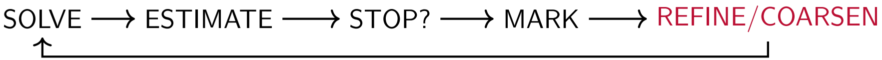

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- Refine/coarsen marked elements

Main results: the norm and the error estimator

- The error norm being estimated:

$$
\begin{aligned}
|||\boldsymbol{v}|||_{sT}^2 := {} & \sum_{\mathcal{K}\in\mathcal{T}_h} \|v\|_{\mathcal{K}}^2 + \sum_{\mathcal{K}\in\mathcal{T}_h} \left\| \left|\beta_s - \tfrac{1}{2}\beta\cdot n\right|^{1/2} [\boldsymbol{v}] \right\|_{\partial\mathcal{K}}^2 + \boldsymbol{T} \sum_{F\in\partial\mathcal{E}_N} \left\| \left|\tfrac{1}{2}\beta\cdot n\right|^{1/2} \mu \right\|_F^2 \\
& + \boldsymbol{T} \sum_{\mathcal{K}\in\mathcal{T}_h} \varepsilon \left\|\overline{\nabla} v\right\|_{\mathcal{K}}^2 + \sum_{\mathcal{K}\in\mathcal{T}_h} \varepsilon h_K^{-1} \left\|[\boldsymbol{v}]\right\|_{\mathcal{Q}_{\mathcal{K}}}^2 + \sum_{\mathcal{K}\in\mathcal{T}_h} \tau_\varepsilon \left\|\partial_t v\right\|_{\mathcal{K}}^2
\end{aligned}
$$

Main results: the norm and the error estimator

- The error estimator:

$$\eta^2 := \sum_{\mathcal{K}\in\mathcal{T}_h} \left(\eta^{\mathcal{K}}\right)^2$$

where

$$\left(\eta^{\mathcal{K}}\right)^2 := \left(\eta_R^{\mathcal{K}}\right)^2 + \sum_{i=1}^{3} \left(\eta_{J,i}^{\mathcal{K}}\right)^2 + \sum_{j=1}^{2} \left(\eta_{BC,j}^{\mathcal{K}}\right)^2$$

where

$$\boldsymbol{\eta_R^{\mathcal{K}} := \lambda_{\mathcal{K}} \, \| R_h^{\mathcal{K}} \|_{\mathcal{K}}} \qquad \eta_{J,1}^{\mathcal{K}} := h_K^{1/2} \varepsilon^{1/2} \left\| [\![\overline{\nabla}_{\overline{n}} u_h]\!] \right\|_{\mathcal{Q}_{\mathcal{K}} \setminus \partial\mathcal{E}}$$

$$\eta_{J,2}^{\mathcal{K}} := \left(\left(\eta_{J,2,1}^{\mathcal{K}}\right)^2 + \left(\eta_{J,2,2}^{\mathcal{K}}\right)^2 \right)^{1/2} \qquad \eta_{J,3}^{\mathcal{K}} := \left(\left(\eta_{J,3,\mathcal{Q}}^{\mathcal{K}}\right)^2 + \left(\eta_{J,3,\mathcal{R}}^{\mathcal{K}}\right)^2 \right)^{1/2}$$

$$\eta_{BC,1}^{\mathcal{K}} := h_K^{1/2} \varepsilon^{-1/2} \, \| R_h^N \|_{\mathcal{Q}_{\mathcal{K}} \cap \partial\mathcal{E}_N} \qquad \eta_{BC,2}^{\mathcal{K}} := \| R_h^N \|_{\partial\mathcal{K} \cap \Omega_0} = \| g - u_h \|_{\partial\mathcal{K} \cap \Omega_0}$$

where

$$\eta_{J,2,1}^{\mathcal{K}} := h_K^{-1/2} \varepsilon^{1/2} \left\| [\boldsymbol{u}_h] \right\|_{\mathcal{Q}_{\mathcal{K}}} \qquad \eta_{J,2,2}^{\mathcal{K}} := h_K^{1/4} \varepsilon^{-1/2} \left\| [\boldsymbol{u}_h] \right\|_{\mathcal{Q}_{\mathcal{K}}}$$

$$\eta_{J,3,\mathcal{Q}}^{\mathcal{K}} := \left\| |\beta_s - \tfrac{1}{2}\beta \cdot n|^{1/2} \, [\boldsymbol{u}_h] \right\|_{\mathcal{Q}_{\mathcal{K}}} \qquad \eta_{J,3,\mathcal{R}}^{\mathcal{K}} := \left\| |\beta_s - \tfrac{1}{2}\beta \cdot n|^{1/2} \, [\boldsymbol{u}_h] \right\|_{\mathcal{R}_{\mathcal{K}}}$$

Main results: reliability and local efficiency

- Reliability:

$$|||\boldsymbol{u} - \boldsymbol{u}_h|||_{sT} \leq cT\varepsilon^{-1/2}\eta$$

- Local efficiency:

$$\eta^{\mathcal{K}} \leq c\left(\sum_{\mathcal{K}\subset\omega_{\mathcal{K}}} \varepsilon^{-1/2}\tilde{\varepsilon}^{-1/2}|||\boldsymbol{u} - \boldsymbol{u}_h|||_{sT,\mathcal{K}} + \mathrm{osc}_h^{\mathcal{K}} + \mathrm{osc}_h^{N}\right)$$

- Local efficiency (*when elements are sufficiently refined*):

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Efficiency index (defined as $\eta/|||\boldsymbol{u}-\boldsymbol{u}_h|||$)

- Pre-asymptotic regime:
  bounded below by $\mathcal{O}(\varepsilon^{1/2})$ and above by $\mathcal{O}(\varepsilon^{-1})$;
- Asymptotic regime:
  bounded below by $\mathcal{O}(\varepsilon^{1/2})$ and above by $\mathcal{O}(\varepsilon^{-1/2})$;

Main results: highlights (and lowlights)

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Saturation assumption

Halving the time-step of every element of $\mathcal{T}_h$, obtaining a subgrid mesh $\mathcal{T}_{\mathfrak{h}}$, we assume there exists $\rho < 1$:

$$\sum_{\mathcal{K}\in\mathcal{T}_h} \tau_\varepsilon \left\|\partial_t(u - u_{\mathfrak{h}})\right\|_{\mathcal{K}}^2 \leq \rho^2 \sum_{\mathcal{K}\in\mathcal{T}_h} \tau_\varepsilon \left\|\partial_t(u - u_h)\right\|_{\mathcal{K}}^2$$

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- The norm of the error, $|||\boldsymbol{u} - \boldsymbol{u}_h|||_{sT}$, is locally computable, as opposed to the popular dual-norm approach
- The norm includes the time derivative, which is estimated using a novel *saturation assumption*
- The localization of the error estimate is both in space and in time (first to the best of our knowledge)
- However:
    - The saturation assumption technique requires $p_t = 1$
    - The reliability analysis requires $\delta t_{\mathcal{K}} = \mathcal{O}(h_K^2)$
    - The analysis assumes a fixed domain

A boundary layer test case

Exact solution:

$$u(t, x_1, x_2) = (1 - \exp(-t)) \left(\frac{\exp((x_1 - 1)/\varepsilon) - 1}{\exp(-1/\varepsilon) - 1} + x_1 - 1 \right) \left(\frac{\exp((x_2 - 1)/\varepsilon) - 1}{\exp(-1/\varepsilon) - 1} + x_2 - 1 \right)$$

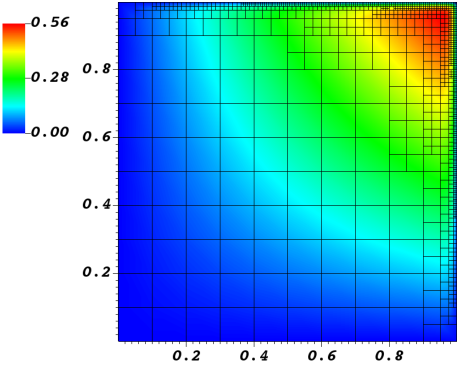


Figure: Solution is for $\varepsilon = 10^{-3}$ and $t = 1.0$

A boundary layer test case

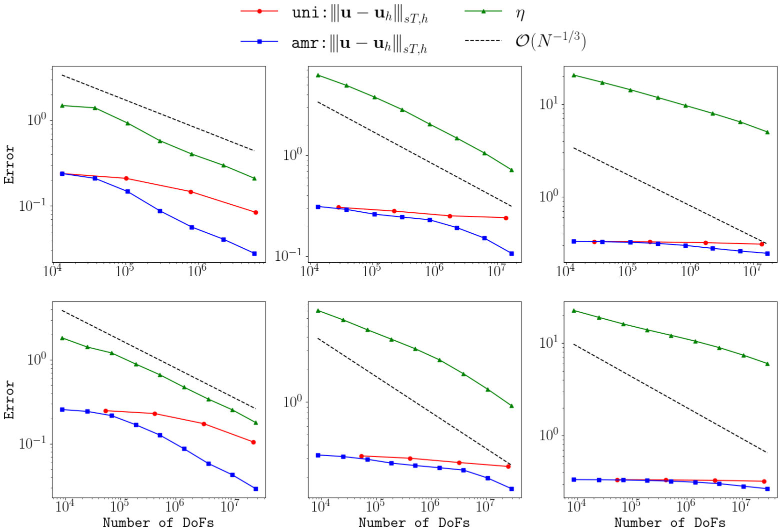


Figure: 1st row: $\delta t_{\mathcal{K}} = h_K$; 2nd row: $\delta t_{\mathcal{K}} = h_K^2$. From left to right: $\varepsilon = 10^{-2}$, 10^{-3}, 10^{-4}.

A boundary layer test case

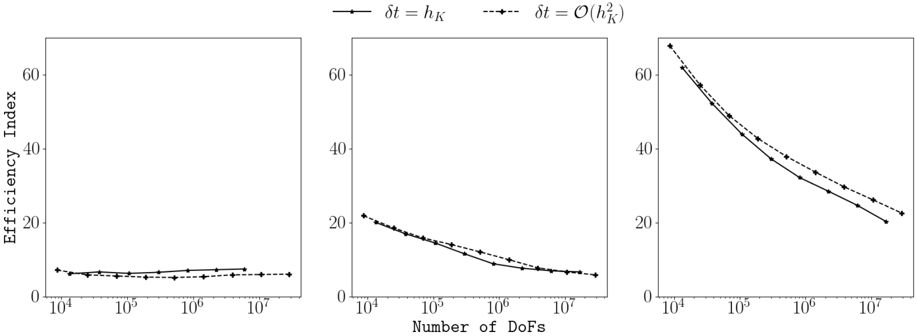
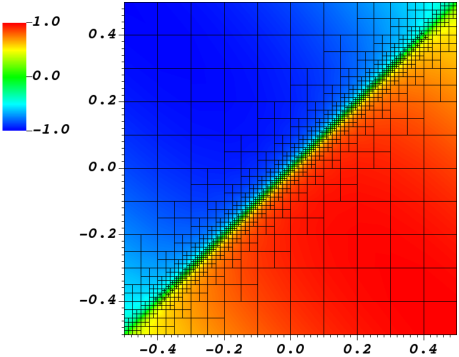


Figure: From left to right: $\varepsilon = 10^{-2}$, 10^{-3}, 10^{-4}.

An interior layer test case

Exact solution: $u(t,x_1,x_2) = (1-\exp(-t))\left(\arctan(\frac{y-x}{\sqrt{2}\varepsilon})\right)\left(1-\frac{(x+y)^2}{2}\right)$



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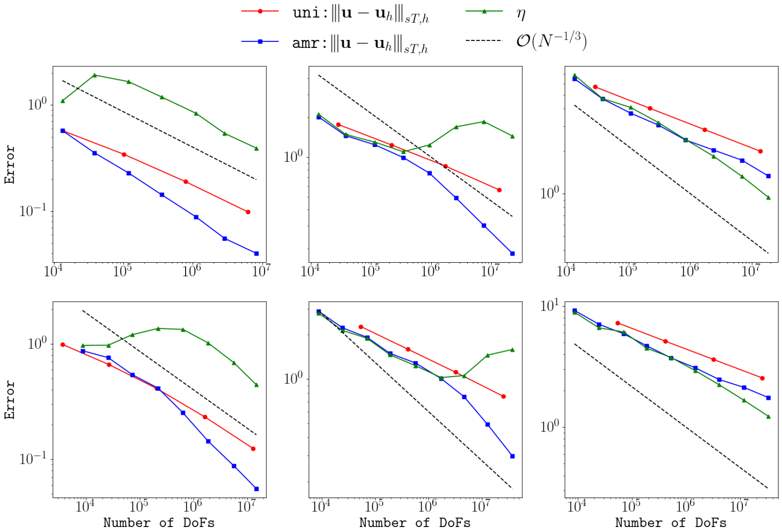

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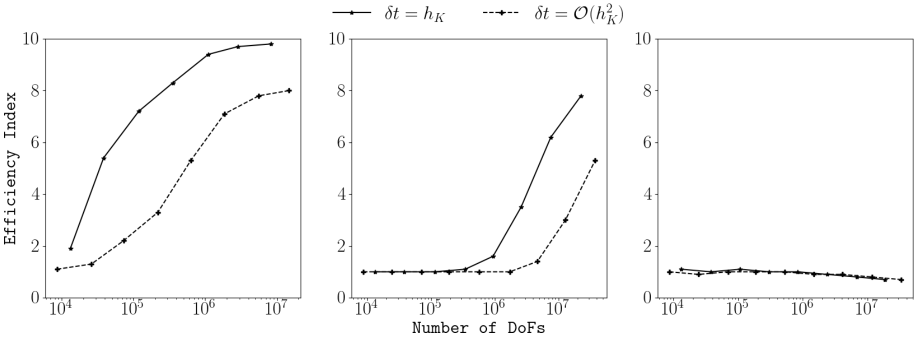

Figure: From left to right: $\varepsilon = 10^{-2}$, 10^{-3}, 10^{-4}.

Thank you!

Find codes, papers and more on my website:

`gregw.xyz`