Space-time HDG for Advection-diffusion on Deforming Domains: the Advection-dominated Regime

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Outline

1. Space-time formulation for advection-diffusion

2. Hybridizable discontinuous Galerkin method

3. A ε-robust *a priori* error estimate

4. Numerical validation

Advection-dominated advection-diffusion problems

The transient problem

$$\partial_t u + \overline{\nabla} \cdot (\bar{\beta} u) - \varepsilon \overline{\nabla}^2 u = f \quad \text{in } \Omega(t),\ 0 < t \leq T$$

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- $\Omega(t) \subset \mathbb{R}^d$: time-dependent polygonal ($d = 2$) or polyhedral ($d = 3$) domain that evolves continuously for $t \in [0, T]$

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- $\beta := (1, \bar{\beta})$: space-time advective field (still divergence-free)
- $\mathcal{E} := \{(t, x) : x \in \Omega(t), 0 < t < T\} \subset \mathbb{R}^{d+1}$:
  the $(d+1)$-dimensional polyhedral space-time domain

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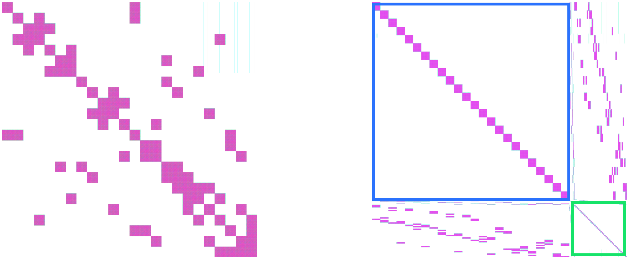

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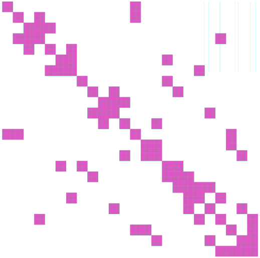

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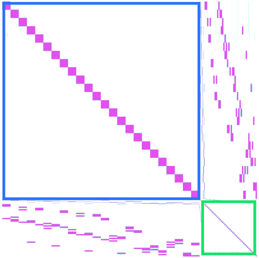

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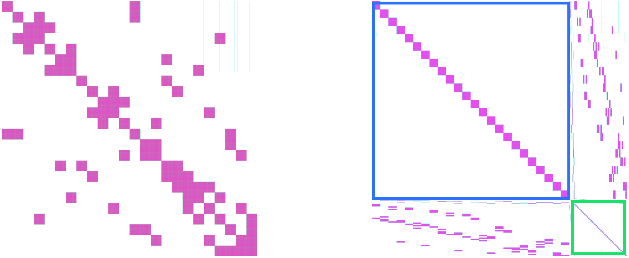
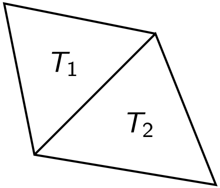


(a) DG dofs:1584

(b) HDG dofs:2046 (=1584+462)

- The global system of HDG has the size of the green box
- The local system in the blue box is easily invertible

HDG: An illustration



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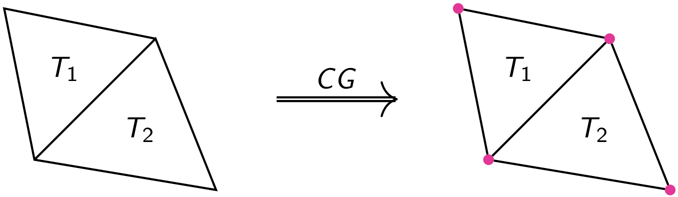

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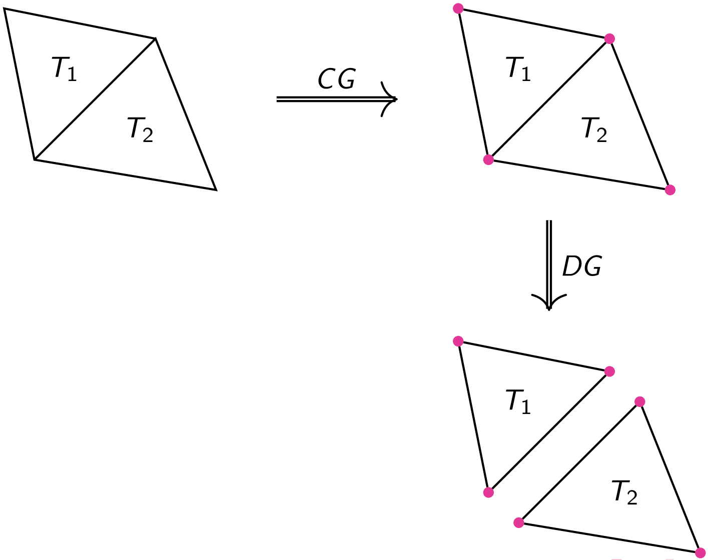

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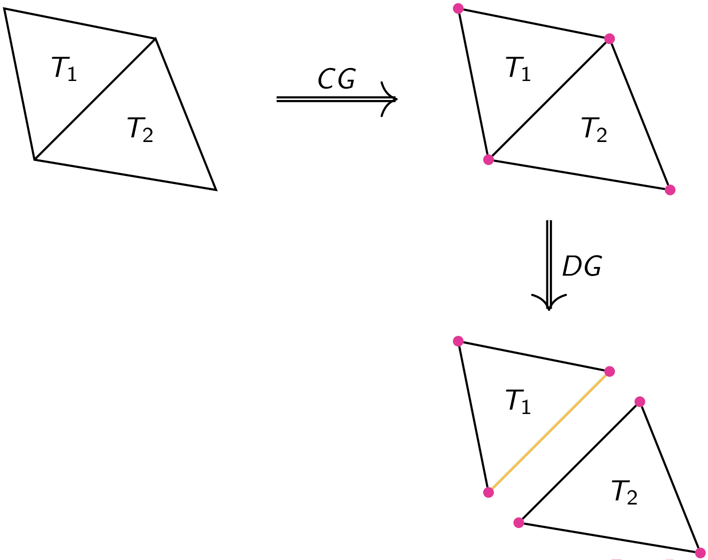

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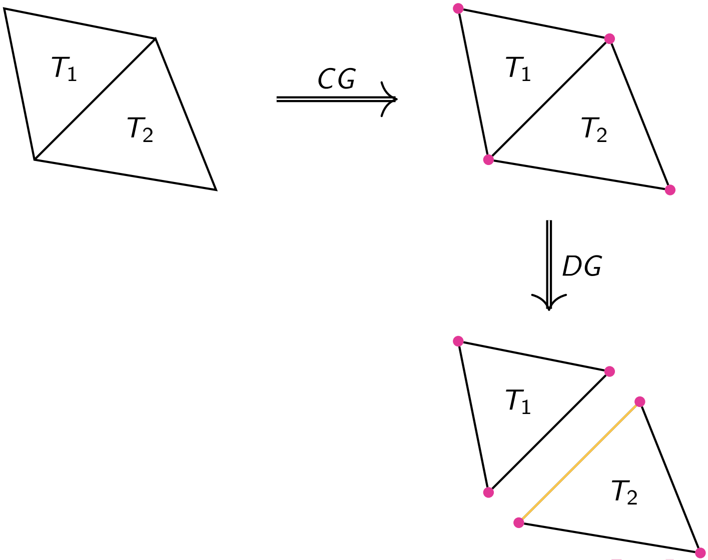

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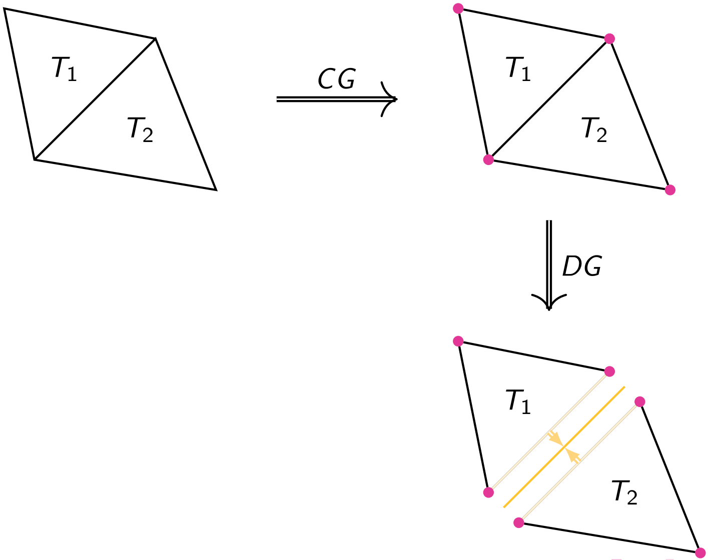

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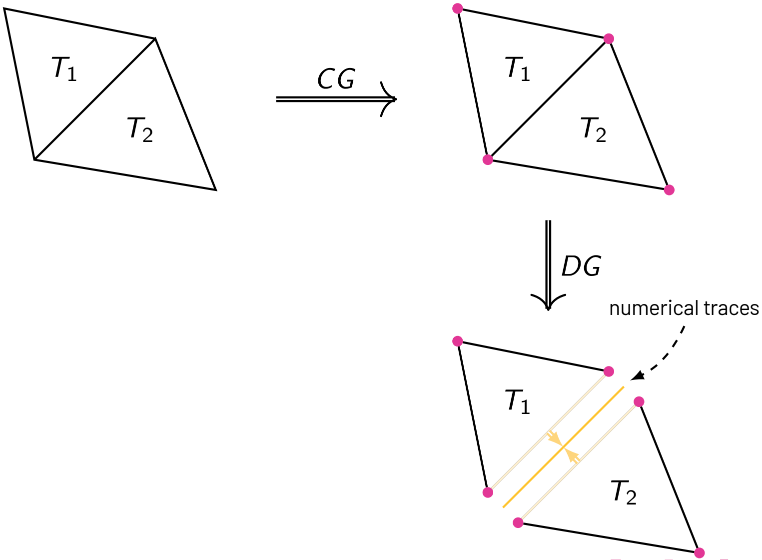

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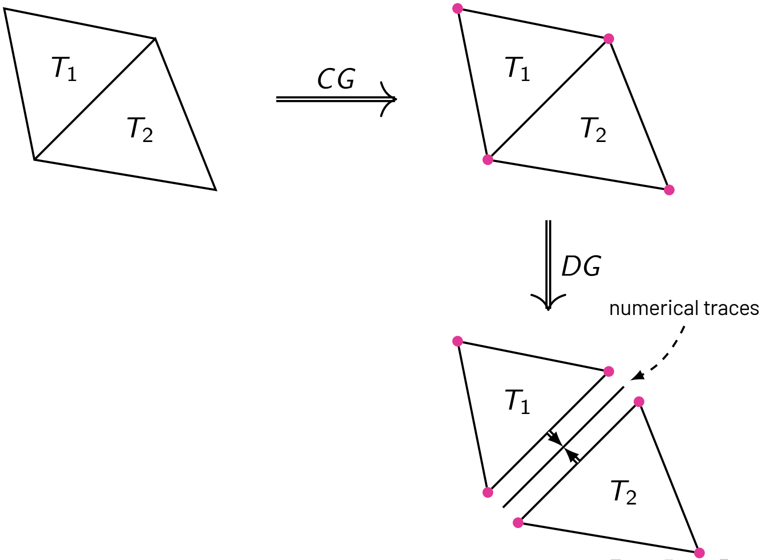

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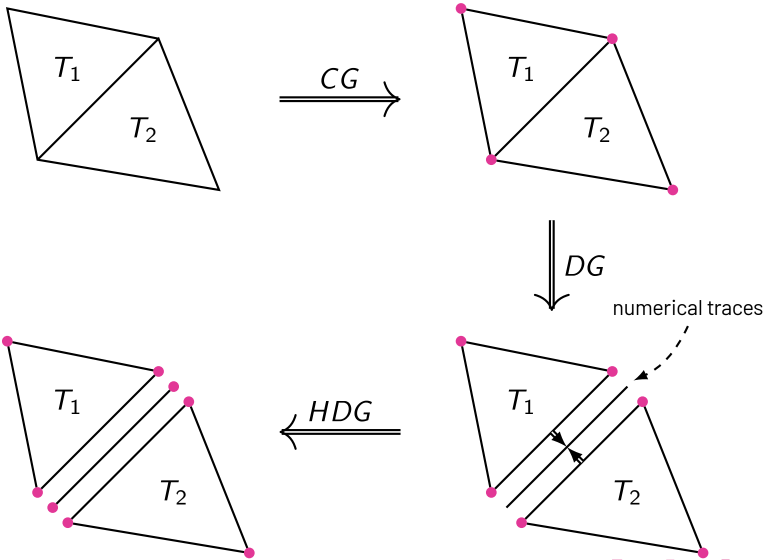

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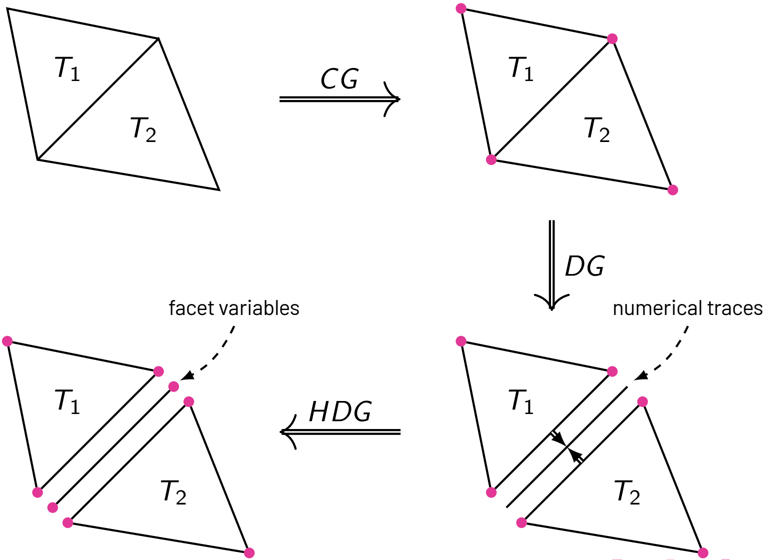

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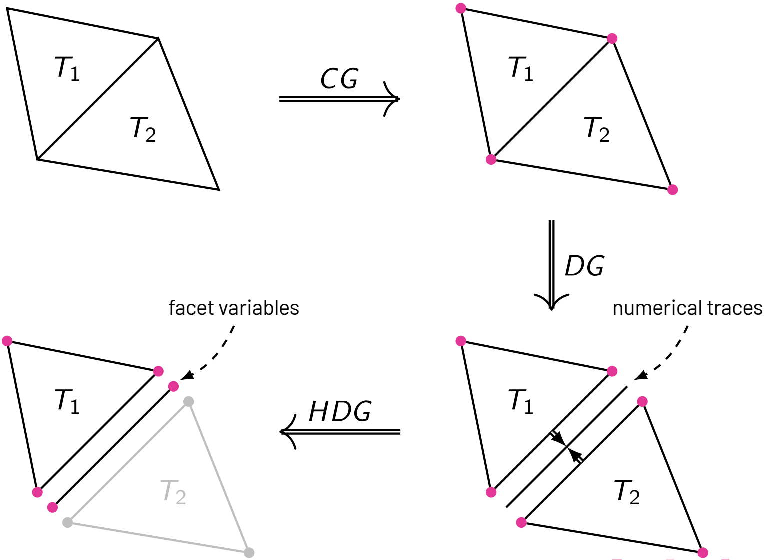

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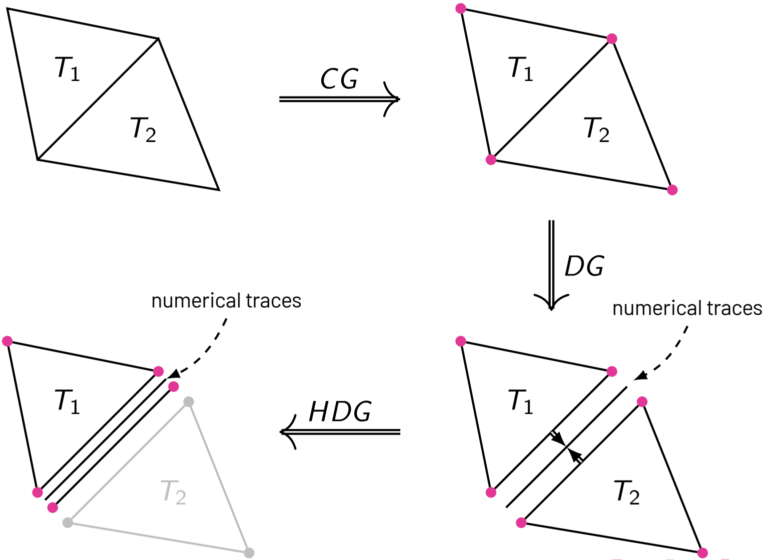

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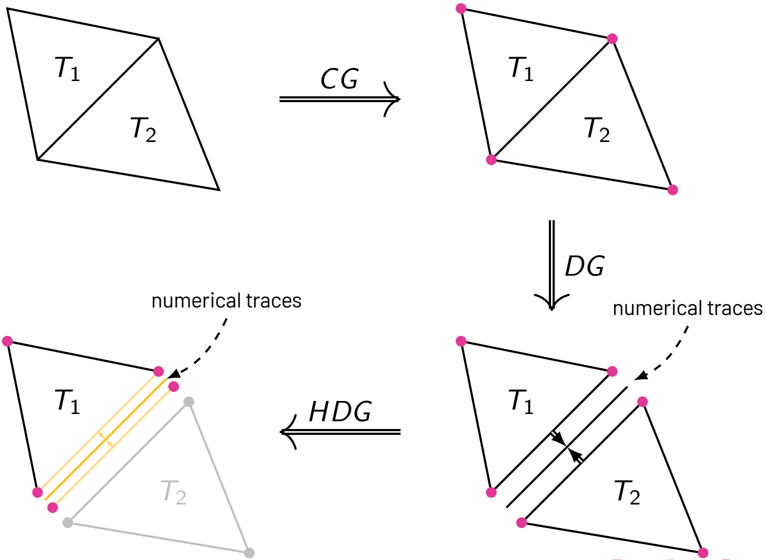

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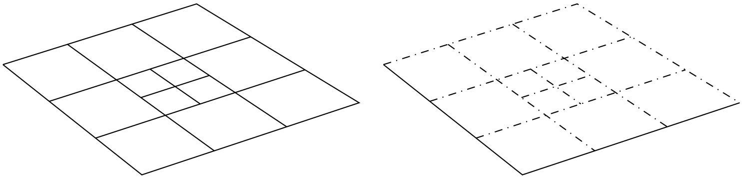

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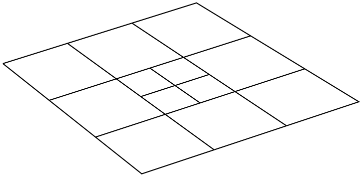

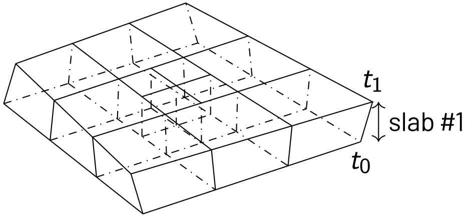

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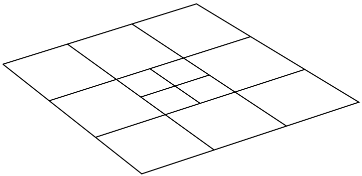

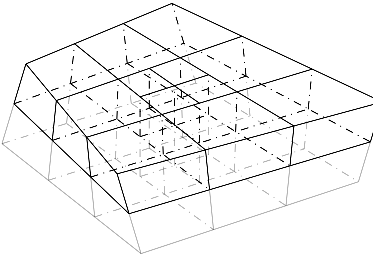

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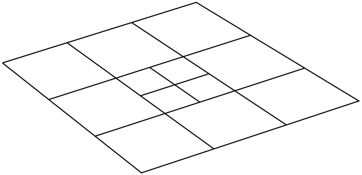
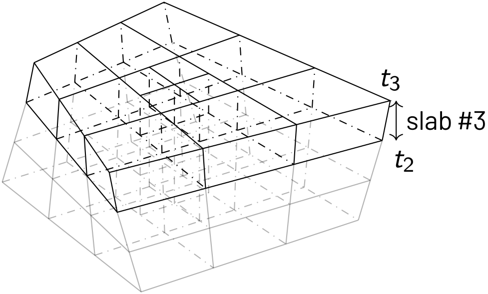
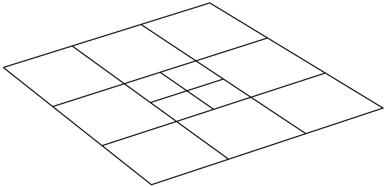
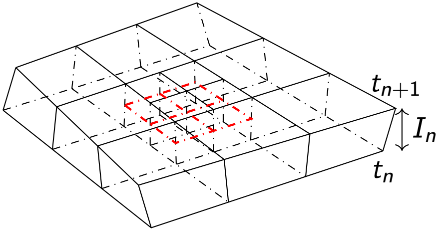




Space-time HDG discretization

- We allow hanging-nodes on both spatial and temporal directions





A space-time HDG discretization

Find $\boldsymbol{u}_h \in \boldsymbol{V}_h$ such that

$$a_h\left(\boldsymbol{u}_h, \boldsymbol{v}_h\right) = (f, v_h)_{\mathcal{T}_h} + \langle g, \mu_h \rangle_{\partial\mathcal{E}_N} \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h,$$

with $a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) := a_{h,d}(\boldsymbol{u}_h, \boldsymbol{v}_h) + a_{h,c}(\boldsymbol{u}_h, \boldsymbol{v}_h)$

$$\begin{aligned}
a_{h,d}\left(\boldsymbol{u}, \boldsymbol{v}\right) :=& \left(\varepsilon \overline{\nabla} u, \overline{\nabla} v\right)_{\mathcal{T}_h} + \left\langle \varepsilon \alpha h_K^{-1} \left[\boldsymbol{u}\right], \left[\boldsymbol{v}\right] \right\rangle_{\mathcal{Q}_h} \\
&- \left\langle \varepsilon \left[\boldsymbol{u}\right], \overline{\nabla}_{\bar{n}} v \right\rangle_{\mathcal{Q}_h} - \left\langle \varepsilon \overline{\nabla}_{\bar{n}} u, \left[\boldsymbol{v}\right] \right\rangle_{\mathcal{Q}_h}, \\
a_{h,c}\left(\boldsymbol{u}, \boldsymbol{v}\right) :=& -\left(\beta u, \nabla v\right)_{\mathcal{T}_h} + \left\langle \zeta^+ \beta \cdot n \lambda, \mu \right\rangle_{\partial\mathcal{E}_N} \\
&+ \left\langle \left(\beta \cdot n\right) \lambda + \beta_s \left[\boldsymbol{u}\right], \left[\boldsymbol{v}\right] \right\rangle_{\partial\mathcal{T}_h}
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- $\alpha = 8p_s^2$: penalty parameter

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- $\beta_s := \sup_{(x,t)\in F} |\beta \cdot n|$: stabilization function[1]
- $\beta_s := \max\{|\beta \cdot n|, 0\}$: classic upwinding
- $\beta_s := \max\left\{\sup_{(x,t)\in F} |\beta \cdot n|, 0\right\}$[2]

[1] G. Fu, W. Qiu, and W. Zhang. *ESAIM:M2AN* 49.1 (2015), pp. 225–256.

[2] Ibid

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- a weighting function technique[4]:

$$\begin{aligned} &\beta \text{ has no closed curves and } \beta(\boldsymbol{x}) \neq 0 \text{ in } \Omega \\ &\Longrightarrow \exists \text{ a weighting function } \varphi \\ &\Longrightarrow |\varphi|\,\|\boldsymbol{e}\| \;\big(\,|\varphi| \text{ is related to the size of the domain}\big) \end{aligned}$$

[3] K. L. A. Kirk et al. *SIAM J. Numer. Anal.* 57.4 (2019), pp. 1677–1696.

[4] B. Ayuso and L. D. Marini. *SIAM J. Numer. Anal.* 47.2 (2009), pp. 1391–1420.

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A ε-robust inf-sup stability

$$|||\boldsymbol{w}_h|||_{ss} \leq c_T \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{a_h(\boldsymbol{w}_h, \boldsymbol{v}_h)}{|||\boldsymbol{v}_h|||_s}$$

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- *ε-robustness: c_T is independent to ε* (but linear to final time T)
- The involved three norms:

$$|||\boldsymbol{v}|||_v^2 := \sum_{\mathcal{K}\in\mathcal{T}_h} \|v\|_{\mathcal{K}}^2 + \sum_{\mathcal{K}\in\mathcal{T}_h} \left\| |\beta_s - \tfrac{1}{2}\beta\cdot n|^{1/2} [\boldsymbol{v}] \right\|_{\partial\mathcal{K}}^2 + \sum_{F\in\partial\mathcal{E}_N} \left\| |\tfrac{1}{2}\beta\cdot n|^{1/2} \mu \right\|_F^2 + \sum_{\mathcal{K}\in\mathcal{T}_h} \varepsilon \|\overline{\nabla} v\|_{\mathcal{K}}^2 + \sum_{\mathcal{K}\in\mathcal{T}_h} \varepsilon h_K^{-1} \|[\boldsymbol{v}]\|_{\mathcal{Q}_{\mathcal{K}}}^2$$

$$|||\boldsymbol{v}|||_s^2 := |||\boldsymbol{v}|||_v^2 + \sum_{\mathcal{K}\in\mathcal{T}_h} \tau_\varepsilon \|\partial_t v\|_{\mathcal{K}}^2 \quad \left(\tau_\varepsilon = \begin{cases} \Delta t_{\mathcal{K}} & \text{when} \quad \delta t_{\mathcal{K}} \le h_K \le \varepsilon \\ \Delta t_{\mathcal{K}} \varepsilon & \text{when} \quad \varepsilon < \delta t_{\mathcal{K}} \le h_K \end{cases} \right)$$

$$|||\boldsymbol{v}|||_{ss}^2 := |||\boldsymbol{v}|||_s^2 + \|v\|_{sd}^2 := |||\boldsymbol{v}|||_s^2 + \sum_{\mathcal{K}\in\mathcal{T}_h} \frac{\delta t_{\mathcal{K}} h_K^2}{\delta t_{\mathcal{K}} + h_K} \|\Pi_h(\beta\cdot\nabla v)\|_{\mathcal{K}}^2$$

A ε-robust inf-sup stability

A ε-robust inf-sup stability

- ε-robust inf-sup w.r.t. $|||\cdot|||_v$:

$$|||\boldsymbol{w}_h|||_v \le c_T \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{a_h(\boldsymbol{w}_h, \boldsymbol{v}_h)}{|||\boldsymbol{v}_h|||_v} \qquad \left(|||\boldsymbol{w}_h|||_v^2 \lesssim \varepsilon^{-1} a_h(\boldsymbol{w}_h, \boldsymbol{w}_h) \right)$$

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  - proved with spatial and temporal hanging-nodes allowed
  - to allow temporal hanging-nodes (local time-stepping), we needed to impose that **ratio between maximum and minimum local time-steps within the same space-time slab is bounded:** $\Delta t_{\mathcal{K}} / \delta t_{\mathcal{K}} < c$

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- ε-robust inf-sup w.r.t. $|||\cdot|||_{ss}$:

$$|||\boldsymbol{w}_h|||_{ss} \leq c_T \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{a_h(\boldsymbol{w}_h, \boldsymbol{v}_h)}{|||\boldsymbol{v}_h|||_s} \left(w/ \ \text{streamline derivative} \ \sum_{\mathcal{K} \in \mathcal{T}_h} \frac{\delta t_{\mathcal{K}} h_K^2}{\delta t_{\mathcal{K}} + h_K} \|\Pi_h (\beta \cdot \nabla v)\|_{\mathcal{K}}^2 \right)$$

A ε-robust error analysis

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$$|||e|||_{ss}^2 \le c_T \left[h^{2p_s}(h + \varepsilon + \tilde{\varepsilon}\delta t) + \delta t^{2p_t+1}(\varepsilon h^{-1} + 1) \right]$$

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This estimate shows

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A ε-robust error analysis

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A ε-robust error analysis

$$|||e|||_{ss}^2 \le c_T\left[h^{2p_s}(h+\varepsilon+\tilde{\varepsilon}\delta t)+\delta t^{2p_t+1}(\varepsilon h^{-1}+1)\right]$$

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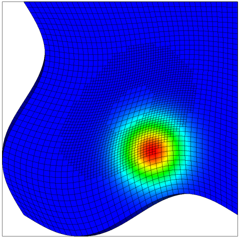
- if $\delta t = h < \varepsilon$

$$|||e|||_{ss} = \mathcal{O}(h^{p_s}+\delta t^{p_t})$$

A $1/2$ drop in the convergence rate is expected after mesh is sufficiently refined.

A rotating Gaussian pulse[5][6]



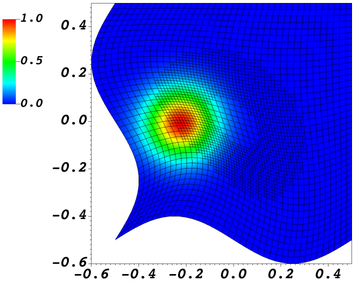
$$\nabla \cdot (\beta u) - \varepsilon \overline{\nabla}^2 u = f \quad \text{in } \mathcal{E}$$

- Data: $\beta = (1, -4x_2, 4x_1)^{\mathsf{T}}, \; f = 0$
- Exact sol: $u(t, x_1, x_2) = \frac{\sigma^2}{\sigma^2 + 2\varepsilon t} \exp\left(-\frac{(\widetilde{x}_1 - x_{1c})^2 + (\widetilde{x}_2 - x_{2c})^2}{2\sigma^2 + 4\varepsilon t}\right)$
- Mesh deformation: $x_i = x_i^u + A\left(\frac{1}{2} - x_i^u\right) \sin\left(2\pi\left(\frac{1}{2} - x_i^* + t\right)\right)$
- Ring of hanging nodes: $|((x_1^c)^2 + (x_2^c)^2)^{1/2} - 0.2| < 0.1$

[5] S. Rhebergen and B. Cockburn. *The Courant–Friedrichs–Lewy (CFL) condition, 80 years after its discovery*. Ed. by C. A. de Moura and C. S. Kubrusly. Birkhäuser Science, 2013, pp. 45–63.

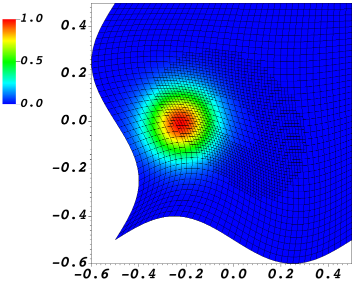
[6] Implemented with the `deal.II` and `PETSc` libraries. Simulated with support provided by Digital Research Alliance of Canada and Math Faculty Computing Facility at the University of Waterloo.

Convergence histories ($\varepsilon = 10^{-2}$)



Cells per slab	Slabs	$p = 1$	Rate	$p = 2$	Rate	$p = 3$	Rate
296	10	4.7e-2	-	7.8e-3	-	1.3e-3	-
1100	20	1.8e-2	1.4	1.6e-3	2.4	1.2e-4	3.6
4372	40	7.7e-3	1.3	3.2e-4	2.3	1.7e-5	3.4
17572	80	3.7e-3	1.1	7.3e-5	2.1	1.4e-6	3.2
70540	160	2.0e-3	0.9	2.3e-5	1.7	2.4e-7	2.4
282580	320	9.0e-4	1.1	4.9e-6	2.2	2.5e-8	3.3

Convergence histories ($\varepsilon = 10^{-8}$)



Cells per slab	Slabs	$p = 1$	Rate	$p = 2$	Rate	$p = 3$	Rate
296	10	1.1e-1	-	1.6e-2	-	2.8e-3	-
1100	20	3.9e-2	1.5	2.8e-3	2.7	2.3e-4	3.8
4372	40	1.1e-2	1.8	4.4e-4	2.7	1.8e-5	3.7
17572	80	3.4e-3	1.7	7.1e-5	2.6	1.4e-6	3.7
70540	160	1.1e-3	1.6	1.2e-5	2.6	1.1e-7	3.6
282580	320	4.0e-4	1.5	2.1e-6	2.5	9.7e-9	3.6

Thank you!